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On the inverse of the directional derivative operator in $\mathbb{R}^N$

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Abstract

In this work, the simplest partial differential equation in $\mathbb{R}^N$ is studied and some of its properties derived. It describes the infinitesimal translation of a function Φ in a fixed direction specified by a unit vector $\mathbf{n}$ and driven by a source function ρ. We work out several forms of the corresponding Green's function and show that the solutions may be viewed as integral transforms of ρ, known as 'divergent beam x-ray transform' in imaging science. Physically, this connection is simply due to straight line propagation of radiation emitted by the spatial source distribution ρ. In particular, we examine the special two-dimensional case to point out the connection with the classical Radon transform. We then show how the Radon transform inversion can be obtained in the context of a complex extension of this equation. Perspectives in higher dimensional space, based on the present approach, are given in the conclusion.

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1. Introduction

In physics, a local conservation law for a vector field $\mathbf{D}$ in $\mathbb{R}^N$ may be expressed as a divergence equation of the form

$$\operatorname{div} \mathbf{D} = \nabla \cdot \mathbf{D} = \rho, \tag{1}$$

where ρ is a given density, acting as a source for $\mathbf{D}$. A few examples of such equations are

- in electrostatics, where $\mathbf{D}$ is the electric charge induction field and ρ a charge distribution density,
- in electromagnetic radiation theory, where $\mathbf{D}$ is the Poynting vector and ρ represents

$$\rho = -\frac{\partial u}{\partial t} - \mathbf{J} \cdot \mathbf{E},$$

here the first term is the time variation of the electromagnetic energy density and the second term is the mechanical work done by the electric field $\mathbf{E}$ on the electric current distribution $\mathbf{J}$,

– in fluid mechanics, where $\mathbf{D}$ is the mass current density, defined as the product of the mass density by the velocity field of the fluid $\mathbf{v}$, and ρ is minus the mass density time rate of change.

Consider now a special case, in which $\mathbf{D}$ is required to have a fixed direction, specified by a unit vector $\mathbf{n}$

$$\mathbf{D} = \mathbf{n}\Phi, \tag{2}$$

where Φ is a scalar flux density. Then the divergence equation (1) takes the form

$$\nabla \cdot \mathbf{D} = \mathbf{n} \cdot \nabla\Phi = \rho, \tag{3}$$

which may be interpreted as the derivative of the flux density Φ in the direction of $\mathbf{n}$.

Φ, as a solution of equation (3) can be thought simply as the 'primitive' of ρ in the $\mathbf{n}$-direction, in other words Φ should be obtained by 'integrating' ρ along the direction of $\mathbf{n}$. This problem appears trivial in one dimension. However in higher dimensional space, some interesting aspects appear and are worth studying. As we shall see later, equation (3) is in fact related to a class of integral transforms in integral geometry in the sense of Gel'fand, which plays an important role in present day imaging science as well as in the field of inverse problems of mathematical physics.

In fact, equation (3) may be viewed as a very special case of the stationary photon transport equation, see e.g. [1]

$$\mathbf{n} \cdot (\nabla\Phi)_{(\mathbf{r},\mathbf{n},E)} = -a(\mathbf{r}, E)\Phi(\mathbf{r}, \mathbf{n}, E) + \rho(\mathbf{r}, \mathbf{n}, E) + Q(\mathbf{r}, \mathbf{n}, E). \tag{4}$$

The first term on the right-hand side of equation (4) represents the loss of photons at site $\mathbf{r}$ and at energy E due to absorption or to scattering into another direction, quantified by $a(\mathbf{r}, E)$, which is nonnegative and represents the rate at which photons are absorbed as they move through the point $\mathbf{r}$ in the direction $\mathbf{n}$. The second term $\rho(\mathbf{r}, \mathbf{n}, E)$ is an emission source term at site $\mathbf{r}$ and energy E. The third term $Q(\mathbf{r}, \mathbf{n}, E)$ accounts for the production of photons due to scattering from all incident photons coming from other directions. It is described by an integral transform with a kernel $\mathbb{K}(\mathbf{n}_0, E_0|\mathbf{n}, E)$ which is nonnegative and proportional to the probability for an incoming photon from the direction $\mathbf{n}_0$ and energy E_0 to be scattered in the direction $\mathbf{n}$ with energy E (it is the rate at which photons moving in the direction $\mathbf{n}_0$ at $\mathbf{r}$ get deflected into the direction $\mathbf{n}$; its physical dimension is an inverse length). This probability is in fact the differential cross-section of the Compton scattering process times the density of scatterers

$$Q(\mathbf{r}, \mathbf{n}, E) = \int_{\mathbb{S}^2} \mathrm{d}\mathbf{n}_0\, \mathbb{K}(\mathbf{n}_0, E_0|\mathbf{n}, E)\Phi_{(\mathbf{r},\mathbf{n}_0,E_0)}; \tag{5}$$

the summation is carried out over the directions of the incident photons $\mathbf{n}_0$. For pure Compton scattering, the outgoing energy E may be the result of many collisions and can be computed via the Compton energy relation. The incident unit vector $\mathbf{n}_0$ is also related to the outgoing propagation unit vector $\mathbf{n}$ by the Compton kinematics.

For now equation (4) has not been fully solved in $\mathbb{R}^N$. In $\mathbb{R}^2$, the works of Novikov [2] and Arbuzov, Bukhgeim and Kazantsev [3, 4] have succeeded to solve the cases with attenuation but without scattering. A nice review has been given by Finch in [5]. Our present work is aimed first at the simplest case, which neglects absorption and scattering as a stepping stone towards the solution in higher dimensions with and without attenuation.

This paper is organized as follows. As boundary conditions are crucial in partial differential equations, we shall construct the Green's function of this operator using Fourier analysis. Then, we compute its explicit form in Cartesian coordinates, which reflects the observations made heuristically before. A second form of the Green's function in spherical coordinates is also presented and coincides with those of Gel'fand in his general construction of Green's functions for the elliptic partial differential equation with constant coefficients [6]. Next we give the explicit solution of equation (3) as line integrals of ρ, which are known under the name of divergent beam x-ray transform in $\mathbb{R}^N$. This name originates from the imaging process in x-ray scanners invented about three decades ago and has ignited an intense activity in related mathematics. Finally, we study the case $N = 2$, where an interesting connection to the $\overline{\partial}$-operator of complex analysis is found as well as with the two-dimensional Radon transform. Conclusion and perspectives on higher dimensions are given in the last section.

2. The solution

2.1. Green's function approach

As it stands, $\mathbf{n} \cdot \nabla$ is just the directional derivative in the direction of the given unit vector $\mathbf{n}$. The solution to (3) is intuitively the indefinite integral or primitive of ρ along the $\mathbf{n}$-direction, to which one may add a function which is constant along the $\mathbf{n}$-direction.

This partial differential equation can be solved by the Green's function method. So if $G(\mathbf{r} - \mathbf{r}')$ is the Green's function solution of

$$\mathbf{n} \cdot \nabla G(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \tag{6}$$

the solution Φ has the expression

$$\Phi(\mathbf{r}) = \int_{\mathbb{R}^N} \mathrm{d}\mathbf{r}'\, G(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}'). \tag{7}$$

Equation (7) may be regarded as a linear integral transform of ρ. The question of recovering ρ in terms of $\Phi(\mathbf{r})$ is a typical inverse problem of interest. Inspection of the one-dimensional case suggests that $G(\mathbf{r}) =$ unit jump at $(\mathbf{r} = \mathbf{r}')$.

Note that under a scaling transformation $\mathbf{r} \to \sigma\mathbf{r}$ for $\sigma \in \mathbb{R}^+$, equation (6) shows that the Green's function transforms as

$$G(\mathbf{r}) = \sigma^{N-1} G(\sigma\mathbf{r}).$$

2.2. The Green's function $G(\mathbf{r})$

As equation (6) is a partial differential equation with constant coefficients, it is most appropriate to work in Fourier space. Following [7], we define the Fourier transform $\widehat{G}(\mathbf{k})$ of $G(\mathbf{r})$ by

$$\widehat{G}(\mathbf{k}) = \int_{\mathbb{R}^N} \mathrm{d}\mathbf{r}\, \mathrm{e}^{-2\pi \mathrm{i}\mathbf{k}\cdot\mathbf{r}} G(\mathbf{r}). \tag{8}$$

Thus conversely, we have

$$G(\mathbf{r}) = \int_{\mathbb{R}^N} \mathrm{d}\mathbf{k}\, \mathrm{e}^{2\pi \mathrm{i}\mathbf{k}\cdot\mathbf{r}} \widehat{G}(\mathbf{k}). \tag{9}$$

Equation (6) shows that

$$2\pi \mathrm{i}(\mathbf{n}\cdot\mathbf{k})\widehat{G}(\mathbf{k}) = 1. \tag{10}$$

Clearly in the sense of distributions, we get

$$\widehat{G}(\mathbf{k}) = \frac{1}{2\pi \mathrm{i}} \left[\frac{1}{(\mathbf{n} \cdot \mathbf{k})} + c\delta(\mathbf{n} \cdot \mathbf{k}) \right]. \tag{11}$$

where c is an arbitrary constant.

Hence $G(\mathbf{r})$ is

$$G(\mathbf{r}) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}^N} \mathrm{d}\mathbf{k}\, \mathrm{e}^{2\mathrm{i}\pi \mathbf{k}\cdot\mathbf{r}} \left[\frac{1}{(\mathbf{n} \cdot \mathbf{k})} + c\delta(\mathbf{n} \cdot \mathbf{k}) \right], \tag{12}$$

which we write as a sum of two terms $G_1(\mathbf{r}) + (c/2\mathrm{i}\pi)G_0(\mathbf{r})$. G_0 is a solution of the homogenous equation, i.e. $\mathbf{n} \cdot \nabla G_0 = 0$, and is of the form $G_0(\mathbf{n}^\perp \cdot \mathbf{r})$.

2.3. Cartesian form of the Green's function

In $\mathbb{R}^N$ let us introduce an orthogonal basis (up to a rotation around $\mathbf{n}$) constructed on $\mathbf{n}$, such that $\mathbf{n}_j^\perp$, $j = 1, \ldots, (N-1)$ are unit vector orthogonal to $\mathbf{n}$ and among themselves. Thus,

$$\mathbf{r} = x_n \mathbf{n} + \sum_{j=1}^{N-1} x_j^\perp \mathbf{n}_j^\perp, \qquad \mathrm{d}\mathbf{r} = \mathrm{d}x_n \prod_{j=1}^{N-1} \mathrm{d}x_j^\perp, \tag{13}$$

$$\mathbf{k} = k_n \mathbf{n} + \sum_{j=1}^{N-1} k_j^\perp \mathbf{n}_j^\perp, \qquad \mathrm{d}\mathbf{k} = \mathrm{d}k_n \prod_{j=1}^{N-1} \mathrm{d}k_j^\perp. \tag{14}$$

Substituting these expressions into (12) yields, for $G(\mathbf{r})$, a product of independent integrals

$$G(\mathbf{r}) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}^N} \mathrm{d}k_n\, \mathrm{e}^{2\mathrm{i}\pi k_n x_n} \left[\frac{1}{k_n} + c\delta(k_n) \right] \prod_{j=1}^{N-1} \mathrm{d}k_j\, \mathrm{e}^{2\mathrm{i}\pi k_j^\perp x_j^\perp}. \tag{15}$$

Using Lavoine's table [7], we have

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \mathrm{d}k_n \frac{\mathrm{e}^{2\mathrm{i}\pi k_n x_n}}{k_n} = \frac{1}{2}\, \mathrm{sgn}(x_n) \tag{16}$$

where $\mathrm{sgn}(x_n)$ is the signum function of $x_n = (\mathbf{n} \cdot \mathbf{r})$. Consequently,

$$G(\mathbf{r}) = \left[\frac{1}{2}\, \mathrm{sgn}(\mathbf{n} \cdot \mathbf{r}) + c \right] \prod_{j=1}^{N-1} \delta\big(\mathbf{r} \cdot \mathbf{n}_j^\perp\big). \tag{17}$$

This form of the solution is in agreement with the fact that $G(\mathbf{r})$ depends on $\big(\mathbf{r} \cdot \mathbf{n}_j^\perp\big)$, but with a unit jump at the origin. We also note that the choice of the orthogonal basis $\big\{\mathbf{n}_j^\perp; j = 1, \ldots, N-1\big\}$ is made up to an orthogonal transformation in $\mathbb{R}^{N-1}$, a subspace orthogonal to $\mathbf{n}$. Hence the Green's function is a distribution and, as it is presented, does not exhibit manifest spherical symmetry in the orthogonal space to $\mathbf{n}$.

2.4. Spherical symmetric form of the Green's function

To display manifest spherical symmetry, a spherically symmetric form for the multidimensional delta function in equation (17) should be used. We derive such a form for $N = 2, 3$, before giving the general expression for arbitrary N.

2.4.1. $N = 2$. The two-dimensional delta function has been the subject of recurrent studies, e.g. [8]. Its Fourier representation is

$$\delta(x_1, x_2) = \delta(x_1)\delta(x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}p\,\mathrm{d}q\,\mathrm{e}^{2\mathrm{i}\pi(k_1x_1+k_2x_2)}. \tag{18}$$

We put in now polar coordinates in $\mathbb{R}^2$ and in its dual, i.e.

$$x_2 = r\cos\theta, \qquad x_1 = r\sin\theta \tag{19}$$

$$k_2 = k\cos\phi, \qquad k_1 = k\sin\phi. \tag{20}$$

Hence formally we obtain

$$\delta(x_1, x_2) = \int_0^{\infty} k\,\mathrm{d}k \int_{-\pi}^{\pi} \mathrm{d}\phi\,\mathrm{e}^{2\mathrm{i}\pi kr\cos(\phi-\theta)} = \int_0^{\infty} k\,\mathrm{d}k\,2\pi J_0(2\pi kr), \tag{21}$$

where $J_0(x)$ is the Bessel function of order zero. The last integral is obviously divergent and should be considered as representing a distribution. But the application of Hankel's identity [9]

$$\frac{1}{r}\delta(r - r') = \int_0^{\infty} k\,\mathrm{d}k\,2\pi J_0(2\pi kr)2\pi J_0(2\pi kr'), \tag{22}$$

for $r' = 0$, and using the fact that $J_0(0) = 1$, yields

$$\delta(x_1, x_2) = \frac{1}{2\pi r}\delta(r). \tag{23}$$

Now recalling a simple scaling property of the delta function, with $r > 0$

$$\delta(r^2) = \frac{1}{2r}\delta(r), \tag{24}$$

we can write

$$\delta(x_1, x_2) = \delta(\pi r^2). \tag{25}$$

Thus, rotational invariance around the polar coordinates at origin O suggests that the area of a circle centered at O can be the variable of the delta function in two dimensions. This notation has a meaning since we are in two dimensions (in one dimension $\delta(x^2)$ has a different meaning [10]).

2.4.2. $N = 3$. As can be expected we will be using a generalized Hankel's identity for spherical Bessel functions $j_l(x)$, which is derived from the previous one with appropriate modifications. Following the same approach, we have

$$\delta(x_1, x_2, x_3) = \delta(x_1)\delta(x_2)\delta(x_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{e}^{2\mathrm{i}\pi(k_1x_1+k_2x_2+k_3x_3)}. \tag{26}$$

We choose now a spherical coordinate system for which $\mathbf{k} = (k_1, k_2, k_3) = k\mathbf{u}$ and $\mathbf{r} = (x_1, x_2, x_3) = r\hat{\mathbf{x}}_3$, so that $(\mathbf{u}\cdot\mathbf{z}) = \cos\gamma$ and $\mathrm{d}\mathbf{k} = k^2\,\mathrm{d}k\,2\pi\sin\gamma\,\mathrm{d}\gamma$:

$$\delta(x_1, x_2, x_3) = \int_0^{\infty} k^2\,\mathrm{d}k \int_0^{\pi} 2\pi\sin\gamma\,\mathrm{d}\gamma\,\mathrm{e}^{2\mathrm{i}\pi kr\cos\gamma}. \tag{27}$$

Angular integration yields $\delta(x_1, x_2, x_3) = \delta(\mathbf{r})$ as

$$\begin{aligned}\delta(\mathbf{r}) &= 4\pi\int_0^{\infty} k^2\,\mathrm{d}k\frac{\sin(2\pi kr)}{2\pi kr} = 4\pi\int_0^{\infty} k^2\,\mathrm{d}k j_0(2\pi kr)\\ &= \frac{4\pi}{8\pi^3}\int_0^{\infty} k'^2\mathrm{d}k'\,j_0(k'r) = \frac{1}{4\pi r^2}\delta(r).\end{aligned} \tag{28}$$

This can be evaluated by the Hankel's identity for spherical Bessel functions, where r' is set to equal 0, i.e.

$$\delta(\mathbf{r}) = \frac{1}{2\pi r^2}\delta(r). \tag{29}$$

But observing that

$$\delta(r^3) = \frac{1}{3r^2}\delta(r), \tag{30}$$

we obtain the expected result

$$\delta(\mathbf{r}) = \delta\left(\frac{4\pi}{3}r^3\right) = \delta(V), \tag{31}$$

where V is the volume of a sphere in three dimensions and manifest rotational invariance in $\mathbb{R}^3$ is explicit.

2.4.3. General N. For general N, we expect to have the representation

$$\delta(\mathbf{r}) = \delta(V_N(r)), \tag{32}$$

where $V_N(r)$ is the volume of the sphere of radius r centered at the origin of coordinates equal to the product of the area of the unit sphere in $\mathbb{R}^N$ times r^N/N

$$V_N(r) = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{r^N}{N} = \frac{\Omega_N}{N}r^N. \tag{33}$$

This result can be proven by extending the Hankel identity to the generalized spherical Bessel function in $\mathbb{R}^N$, as explained in appendix A.

Let $\mathbf{r}_\perp$ be the projection of $\mathbf{r}$ onto the subspace orthogonal to $\mathbf{n}$. If $r_\perp = |\mathbf{r}_\perp|$, then we have

$$r_\perp^2 = [(\mathbf{r}\cdot\mathbf{r}) - (\mathbf{r}\cdot\mathbf{n})^2]. \tag{34}$$

Then the delta function in the subspace of dimension $(N-1)$ orthogonal to $\mathbf{n}$ has a manifest rotational invariant expression

$$\delta(\mathbf{r}_\perp) = \delta\left(\frac{\Omega_{N-1}}{N-1}[(\mathbf{r}\cdot\mathbf{r}) - (\mathbf{r}\cdot\mathbf{n})^2]^{\frac{N-1}{2}}\right). \tag{35}$$

Consequently the Green's function has the form function

$$G(\mathbf{r}) = \left[\frac{1}{2}\,\mathrm{sgn}(\mathbf{n}\cdot\mathbf{r}) + c\right]\delta\left(\frac{\Omega_{N-1}}{N-1}[(\mathbf{r}\cdot\mathbf{r}) - (\mathbf{r}\cdot\mathbf{n})^2]^{\frac{N-1}{2}}\right). \tag{36}$$

2.5. Integral representation of the Green's function over the unit sphere in dual space $\mathbb{R}^N$

An alternative explicit form of the Green's function can be given using spherical coordinates in Fourier space. Let $\mathbf{k} = k\widetilde{\omega}$ where $\widetilde{\omega}$ is the unit vector of $\mathbf{k}$, then we have

$$\mathrm{d}\mathbf{k} = k^{N-1}\,\mathrm{d}k\,\mathrm{d}\widetilde{\omega}, \tag{37}$$

where $\mathrm{d}\widetilde{\omega}$ is the area element on the unit sphere in $\mathbb{R}^{N-1}$. Hence

$$G(\mathbf{r}) = \frac{1}{2\pi\mathrm{i}}\int_0^\infty\int_{\mathbb{S}^{N-1}} k^{N-1}\mathrm{d}k\,\mathrm{d}\widetilde{\omega}\,\mathrm{e}^{2\mathrm{i}\pi k(\widetilde{\omega}\cdot\mathbf{r})}\left[\frac{1}{k(\widetilde{\omega}\cdot\mathbf{n})} + \frac{c}{k}\delta(\widetilde{\omega}\cdot\mathbf{n})\right]. \tag{38}$$

Performing first the k-integration using Lavoine's table [7]

$$\int_0^\infty k^{N-2}\,\mathrm{d}k\,\mathrm{e}^{2\mathrm{i}\pi k(\widetilde{\omega}\cdot\mathbf{r})} = \left(\frac{1}{2\mathrm{i}\pi}\right)^{N-1}\left[\frac{(N-2)!}{(-\widetilde{\omega}\cdot\mathbf{r})^{N-1}} + \mathrm{i}\pi(-1)^{N-2}\delta^{(N-2)}(-\widetilde{\omega}\cdot\mathbf{r})\right], \tag{39}$$

we end up with an integral on the unit sphere in $\mathbb{R}^N$

$$G(\mathbf{r}) = \left(\frac{1}{2\mathrm{i}\pi}\right)^N \int_{\mathbb{S}^{N-1}} \mathrm{d}\widetilde{\omega}\left[\frac{(N-2)!}{(-\widetilde{\omega}\cdot\mathbf{r})^{N-1}} + \mathrm{i}\pi(-1)^N\delta^{(N-2)}(-\widetilde{\omega}\cdot\mathbf{r})\right]\left[\frac{1}{(\widetilde{\omega}\cdot\mathbf{n})} + c\delta(\widetilde{\omega}\cdot\mathbf{n})\right]. \tag{40}$$

Here $\delta^{(N-2)}(x)$ is the $(N-2)$th derivative of the δ-function with respect to its argument. The right-hand side of equation (40) is a sum of two terms. For given N, one term is real and the other imaginary. Thus, as $G(\mathbf{r})$ is a real valued function, only the real term is the correct expression, the other should vanish.

For $c = 0$, we have explicitly:
with $N = 2m$

$$G_1(\mathbf{r}) = \frac{(-1)^{m-1}(2m-2)!}{(2\pi)^{2m}}\int_{\mathbb{S}^{2m-1}} \mathrm{d}\widetilde{\omega}\frac{1}{(\widetilde{\omega}\cdot\mathbf{r})^{2m-1}(\widetilde{\omega}\cdot\mathbf{n})}, \tag{41}$$

and with $N = (2m+1)$

$$G_1(\mathbf{r}) = \frac{(-1)^{m-1}}{2(2\pi)^{2m+1}}\int_{\mathbb{S}^{2m}} \mathrm{d}\widetilde{\omega}\frac{\delta^{(2m-1)}(-\widetilde{\omega}\cdot\mathbf{r})}{(\widetilde{\omega}\cdot\mathbf{n})}. \tag{42}$$

These are precisely the expressions one can derive from formulae established by Gel'fand for the Green's functions of general constant coefficients elliptic partial differential operators [6].

As an illustration, we examine the two simplest cases, i.e. $N = 2, 3$.

For $N = 2$,

$$G_1(\mathbf{r}) = -\frac{1}{4\pi^2}\int_{\mathbb{S}^1} \mathrm{d}\widetilde{\omega}\frac{1}{(\widetilde{\omega}\cdot\mathbf{r})(\widetilde{\omega}\cdot\mathbf{n})}, \tag{43}$$

we can also explicitly show that the imaginary part is 0, i.e.

$$-\frac{\mathrm{i}}{4\pi}\int_{\mathbb{S}^1} \mathrm{d}\widetilde{\omega}\frac{\delta(-\widetilde{\omega}\cdot\mathbf{r})}{(\widetilde{\omega}\cdot\mathbf{n})} = 0. \tag{44}$$

For $N = 3$,

$$G_1(\mathbf{r}) = \frac{1}{8\pi^2}\int_{\mathbb{S}^2} \mathrm{d}\widetilde{\omega}\frac{\delta'(-\widetilde{\omega}\cdot\mathbf{r})}{(\omega\cdot\mathbf{n})}, \tag{45}$$

and the imaginary part can be shown to be 0 by computation, i.e.

$$\frac{\mathrm{i}}{8\pi^2}\int_{\mathbb{S}^2} \mathrm{d}\widetilde{\omega}\frac{1}{(\widetilde{\omega}\cdot\mathbf{r})^2(\widetilde{\omega}\cdot\mathbf{n})} = 0. \tag{46}$$

As for the G_0, we compute the real parts (proportional to c) of equation (40). Thus with $N = 2m$

$$G_0(\mathbf{r}) = \frac{(-1)^m(2m-2)!}{(2\pi)^{2m}}\int_{\mathbb{S}^{2m-1}} \mathrm{d}\widetilde{\omega}\frac{1}{(-\widetilde{\omega}\cdot\mathbf{r})^{2m-1}}\delta(\widetilde{\omega}\cdot\mathbf{n}), \tag{47}$$

and with $N = (2m+1)$

$$G_0(\mathbf{r}) = \frac{(-1)^{m+1}}{2(2\pi)^{2m}}\int_{\mathbb{S}^{2m}} \mathrm{d}\widetilde{\omega}\,\delta^{(2m-1)}(-\widetilde{\omega}\cdot\mathbf{r})\delta(\widetilde{\omega}\cdot\mathbf{n}). \tag{48}$$

These integrals should vanish, as one expects them to be the solution of the homogeneous equation. Explicit calculations can be performed in low dimensions, i.e.

for $N = 2$,

$$G_0(\mathbf{r}) = \frac{1}{4\pi^2}\int_{\mathbb{S}^1} \mathrm{d}\widetilde{\omega}\frac{1}{(\widetilde{\omega}\cdot\mathbf{r})}\delta(\widetilde{\omega}\cdot\mathbf{n}) = 0, \tag{49}$$

and for $N = 3$,

$$G_0(\mathbf{r}) = -\frac{1}{8\pi^2}\int_{\mathbb{S}^2} \mathrm{d}\widetilde{\omega}\delta'(\widetilde{\omega}\cdot\mathbf{r})\delta(\widetilde{\omega}\cdot\mathbf{n}) = 0. \tag{50}$$

where the proof is obtained as follows: considering $\psi(r,\theta,\phi)$ as a test function and supposing $\mathbf{n}$ in the z-direction, $\widetilde{\omega} \equiv (1,\theta,\phi)$ and $\mathbf{r} \equiv (r,\alpha,\beta)$

$$\begin{aligned}
\mathfrak{I} &= \frac{1}{8\pi^2}\int_{\mathbb{S}^2} \mathrm{d}\widetilde{\omega}\psi(r,\theta,\phi)\delta'(\widetilde{\omega}\cdot\mathbf{r})\delta(\widetilde{\omega}\cdot\mathbf{n}) \\
&= \frac{1}{8\pi^2}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\psi(r,\theta,\phi)\delta'(r(\cos\theta\cos\alpha+\sin\theta\sin\alpha\cos(\beta-\phi)))\delta(\cos\theta) \\
&= \frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\phi\psi\left(r,\frac{\pi}{2},\phi\right)\delta'(r\sin\alpha\cos(\beta-\phi)) \\
&= \frac{1}{2}\int_0^{2\pi}\mathrm{d}\phi\psi\left(r,\frac{\pi}{2},\phi\right)\delta'(r\sin\alpha\cos(\beta-\phi)) = \frac{\psi'\left(r,\frac{\pi}{2},\frac{\pi}{2}\right)}{2r^2\sin^2\alpha}
\end{aligned} \tag{51}$$

in equation (50), the test function is just 1 thus, $G_0(\mathbf{r})$ is equal to 0.

2.6. Integral representation of the Green's function in $\mathbb{R}^N$

Equation(12) can be transformed into a ray representation via the Fourier representation of

$$\frac{1}{2\pi\mathrm{i}(\mathbf{n}\cdot\mathbf{k})} = \int_{-\infty}^{\infty}\mathrm{d}s\,\mathrm{e}^{-2\pi\mathrm{i}s(\mathbf{k}\cdot\mathbf{n})}\left(\frac{1}{2}\,\mathrm{sgn}(s)\right). \tag{52}$$

In fact, we have after exchanging integration order

$$\begin{aligned}
G_1(\mathbf{r}) &= \int_{-\infty}^{\infty}\mathrm{d}s\left(\frac{1}{2}\,\mathrm{sgn}(s)+c\right)\int_{\mathbb{R}^N}\mathrm{d}\mathbf{k}\,\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}(\mathbf{r}-s\mathbf{n})}, \\
&= \left(c+\frac{1}{2}\right)\int_0^{\infty}\mathrm{d}s\,\delta(\mathbf{r}-s\mathbf{n}) + \left(c-\frac{1}{2}\right)\int_0^{\infty}\mathrm{d}s\,\delta(\mathbf{r}+s\mathbf{n}).
\end{aligned} \tag{53}$$

We can verify that each integral of equation (55) is separately a solution of equation (6).

Observe that from the definition of $G_0(\mathbf{r})$, given in equation (12) we have

$$G_0(\mathbf{r}) = \int_{\mathbb{R}^N}\mathrm{d}\mathbf{k}\,\mathrm{e}^{2\mathrm{i}\pi\mathbf{k}\cdot\mathbf{r}}\delta(\mathbf{n}\cdot\mathbf{r}) = \int_{-\infty}^{\infty}\mathrm{d}s\,\delta(\mathbf{r}-s\mathbf{n}). \tag{54}$$

2.7. Solution of the problem as divergent beam x-ray transform

The solution of equation (3) using $G(\mathbf{r})$ can now be given as a linear combination of integrals of $\rho(\mathbf{r})$

$$\Phi(\mathbf{r},\mathbf{n}) = \left(c+\frac{1}{2}\right)\int_0^{\infty}\mathrm{d}s\,\rho(\mathbf{r}-s\mathbf{n}) + \left(c-\frac{1}{2}\right)\int_0^{\infty}\mathrm{d}s\,\rho(\mathbf{r}+s\mathbf{n}), \tag{55}$$

along the directions of $\mathbf{n}$ and $-\mathbf{n}$ in $\mathbb{R}^N$. In fact each of these two integrals are also solutions

$$\int_0^{\infty}\mathrm{d}s\,\rho(\mathbf{r}+s\mathbf{n}), \qquad \text{resp.} \qquad \int_0^{\infty}\mathrm{d}s\,\rho(\mathbf{r}-s\mathbf{n}) \tag{56}$$

provided that $\rho(|\mathbf{r}| \to \infty) = 0$ (resp. $\rho(|\mathbf{r}| \to -\infty) = 0$).

Since $\Phi(\mathbf{r}, \mathbf{n})$ depends on $\mathbf{r}$ and on $\mathbf{n}$, or $(2N-1)$ parameters, equation (56) can only be considered as the definition of an integral transform, if $(N-1)$ necessary conditions are imposed before an inversion procedure is contemplated. Of course the previous parametrization is redundant. It is sufficient to give $\mathbf{n}(N-1)$ parameters and then consider inside the hyperplane orthogonal to $\mathbf{n}$ containing the coordinate origin O a point of the chosen line, with $(N-1)$ parameters. So the net number of independent parameters is $2(N-1)$. To reconstruct a function on $\mathbb{R}^N$, we have several possibilities depending on how we restrict $\mathbf{n}$ and $\mathbf{r}$ in space such that the total number of variables is N. If $\mathbf{n}$ is unrestricted on the unit sphere, then $\mathbf{r}$ must be on a curve, i.e. $\mathbf{r}(t)$ with $t \in \mathbb{R}$. This is precisely the divergent beam x-ray transformation (or cone-beam transform) in $\mathbb{R}^N$, see [11], [12]. In particular, for $\mathbb{R}^3$, it is known that one must require that $\mathbf{r} = \mathbf{r}(t)$ or that the point source describes a conditioned space curve [13] to have an inverse transform. If $\mathbf{n}$ belongs to a circle of the unit sphere (intersection of a hyperplane and the unit sphere), then $\mathbf{r}$ can describe a hypersurface, i.e. $\mathbf{r}(u, v)$., etc. This important topic will be tackled in a future work.

3. The special case $N = 2$ and its complex extension

In two dimensions, the situation is interesting since the labeling of the line integral by $\mathbf{r}$ and the angle θ for the unit vector $\mathbf{n}$ is redundant in a coordinate system. It is sufficient to give the distance from the coordinate origin to the line and θ. Then this x-ray transform is equivalent to the Radon transform in $\mathbb{R}^2$. In this section, we examine its relation to this simple partial differential equation.

3.1. Green's functions of the complex formulation

As shown earlier, the Green's function in $\mathbb{R}^2$ (without the homogeneous part) is $G_1(\mathbf{r})$ given by a distribution in two dimensions

$$G_1(\mathbf{r}) = \tfrac{1}{2}\,\mathrm{sgn}(\mathbf{r}\cdot\mathbf{n})\delta(\mathbf{r}\cdot\mathbf{n}^{\perp}). \tag{57}$$

Here once $\mathbf{n}$ is given, $\mathbf{n}^{\perp}$ is uniquely defined, if the positive rotation direction in $\mathbb{R}^2$ is specified. In a Cartesian coordinates system, with $\mathbf{r} = (x, y), \mathbf{n} = (\cos\theta, \sin\theta)$ and $\mathbf{n}^{\perp} = (-\sin\theta, \cos\theta)$, the directional derivative operator $\mathbf{n}\cdot\nabla$ is explicitly

$$\mathbf{n}\cdot\nabla = \left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right), \tag{58}$$

and introducing the complex notations $(z = x + \mathrm{i}y, \bar{z} = x - \mathrm{i}y)$, it writes as

$$\mathbf{n}\cdot\nabla = \left(\mathrm{e}^{\mathrm{i}\theta}\frac{\partial}{\partial z} + \mathrm{e}^{-\mathrm{i}\theta}\frac{\partial}{\partial \bar{z}}\right). \tag{59}$$

In previous sections, we have already found the Green's function G_1 of this operator, i.e. $\mathbf{n}\cdot\nabla G_1 = \delta(x, y)$ and established that it is a product of two one-dimensional distributions along orthogonal directions. A natural question to ask is whether or not this Green's function can be represented as boundary values of analytic functions with respect to a complex parameter on some contour in $\mathbb{C}$. In the original form, the real parameter here is, of course, the angle θ of the unit vector $\mathbf{n}$, which has appeared in equation (59) as a unimodular complex number $\mathrm{e}^{\mathrm{i}\theta}$.

In the following, we shall extend it to a complex parameter $\lambda = |\lambda|\,\mathrm{e}^{\mathrm{i}\theta}$ and introduce the 'extended' form of the $\mathbf{n}\cdot\nabla$ operator which shall be called ∇_λ

$$\nabla_\lambda = \left(|\lambda|\,\mathrm{e}^{\mathrm{i}\theta}\frac{\partial}{\partial z} + \frac{1}{|\lambda|\,\mathrm{e}^{\mathrm{i}\theta}}\frac{\partial}{\partial \bar{z}}\right) = \left(\lambda\frac{\partial}{\partial z} + \frac{1}{\lambda}\frac{\partial}{\partial \bar{z}}\right). \tag{60}$$

Under this form both derivatives with respect to z and to $\overline{z}$ appear. We may ask whether a choice of a new variable can eliminate, for example, one of them and turns ∇_λ operator into a Cauchy–Riemann operator, or $\overline{\partial}$-operator.

Let us now consider the new complex variable

$$\zeta = \frac{l}{\mathrm{i}}(\lambda^{-1}z - \lambda\overline{z}), \tag{61}$$

where l is a real number. Its expression in terms of the original vectors $(\mathbf{r}, \mathbf{n}, \mathbf{n}^\perp)$ is particularly simple

$$\zeta = l[(|\lambda| + |\lambda|^{-1})(\mathbf{r}\cdot\mathbf{n}^\perp) + \mathrm{i}(|\lambda| - |\lambda|^{-1})(\mathbf{r}\cdot\mathbf{n})], \tag{62}$$

displaying clearly real and imaginary parts.

Now, under this change of complex variable, the operator ∇_λ of equation (60), transforms into

$$\nabla_\lambda = \frac{l}{\mathrm{i}}(|\lambda|^2 - |\lambda|^{-2})\frac{\partial}{\partial\overline{\zeta}}. \tag{63}$$

What is readily known about the so-called $\overline{\partial}$-operator is that its elementary solution, (see [14]) for a complex variable $z = x + \mathrm{i}y$, is $1/(\pi z)$ (see appendix B), i.e.

$$\frac{\partial}{\partial\overline{z}}\frac{1}{\pi z} = \delta(x, y). \tag{64}$$

Recall that x and y are real Cartesian coordinates and that $\delta(x, y) = \delta(x)\delta(y)$. This result can be derived, for example, from the elementary solution of the Laplace operator in two dimensions. We thus conclude that

$$\frac{\partial}{\partial\overline{\zeta}}\frac{1}{\pi\zeta} = \delta(l(|\lambda| + |\lambda|^{-1})(\mathbf{r}\cdot\mathbf{n}^\perp), l(|\lambda| - |\lambda|^{-1})(\mathbf{r}\cdot\mathbf{n})), \tag{65}$$

which, using the scaling property of the one-dimensional delta function, can be rewritten as

$$\frac{\partial}{\partial\overline{\zeta}}\frac{1}{\pi\zeta} = \frac{1}{l^2||\lambda|^2 - |\lambda|^{-2}|}\delta(\mathbf{r}\cdot\mathbf{n}^\perp, \mathbf{r}\cdot\mathbf{n}). \tag{66}$$

Hence by multiplying this equation on both sides by

$$\frac{l}{\mathrm{i}}(|\lambda|^2 - |\lambda|^{-2}) \tag{67}$$

and using

$$\mathrm{sgn}(|\lambda|^2 - |\lambda|^{-2}) = \frac{(|\lambda|^2 - |\lambda|^{-2})}{\left||\lambda|^2 - |\lambda|^{-2}\right|} = \mathrm{sgn}(|\lambda| - 1), \tag{68}$$

we obtain

$$\nabla_\lambda\left(\frac{1}{\pi\zeta}\right) = \frac{1}{\mathrm{i}l}\mathrm{sgn}(|\lambda| - 1)\delta(\mathbf{r}\cdot\mathbf{n}, \mathbf{r}\cdot\mathbf{n}^\perp). \tag{69}$$

But as the transformation from the real variables $(s = (\mathbf{r}\cdot\mathbf{n}^\perp), t = (\mathbf{r}\cdot\mathbf{n}))$ to the variables (x, y) is just a rotation of angle θ, the Jacobian determinant being 1, we can write

$$\delta((\mathbf{r}\cdot\mathbf{n}), (\mathbf{r}\cdot\mathbf{n}^\perp) = \delta(x, y). \tag{70}$$

Hence

$$\nabla_\lambda\left(\frac{\mathrm{i}l\,\mathrm{sgn}(|\lambda| - 1)}{\pi\zeta}\right) = \delta(x, y), \tag{71}$$

and the elementary solution of the operator ∇_λ is

$$\widetilde{G}_\lambda(\zeta) = \left(\frac{\mathrm{i}l\ \mathrm{sgn}(|\lambda|-1)}{\pi\zeta}\right) = \frac{\mathrm{sgn}(|\lambda|-1)}{\pi(\lambda\bar{z}-\lambda^{-1}z)}, \tag{72}$$

a result found by [1]. As can be observed, this expression is independent of l after the substitution of the expression of ζ from equation (61) and explicitly we have

$$\widetilde{G}(\mathbf{r}\cdot\mathbf{n}, \mathbf{r}\cdot\mathbf{n}^\perp, \lambda) = \frac{\mathrm{i}}{2\pi}\frac{\mathrm{sgn}(|\lambda|-1)}{\left[\left(\frac{|\lambda|+|\lambda|^{-1}}{2}\right)(\mathbf{r}\cdot\mathbf{n}^\perp) + \mathrm{i}\left(\frac{|\lambda|-|\lambda|^{-1}}{2}\right)(\mathbf{r}\cdot\mathbf{n})\right]}. \tag{73}$$

Thus we have two elementary solutions depending on whether $|\lambda|$ is larger or smaller than 1. Observe also apart from this sign factor $\mathrm{sgn}(|\lambda|-1)$, it has the correct form of the analytical representation of a delta function (see [15])

$$\widetilde{\delta}(z) = \frac{1}{2\mathrm{i}\pi}\int_{-\infty}^{\infty}\mathrm{d}t\,\frac{\delta(t)}{(t-z)} = \frac{\mathrm{i}}{2}\frac{1}{\pi z}. \tag{74}$$

We now look at the limiting cases for which $|\lambda| \to 1$ to see whether or not the found $\widehat{G}$ yields the constructed solution $G_1(\mathbf{r})$ of equation (57). In the two limiting cases of, we can see that using the so-called Plemelj formulae [15]

$$\lim_{\epsilon\to 0^+}\frac{1}{x\pm\mathrm{i}\epsilon} = \mathrm{P.V.}\frac{1}{x} \mp \mathrm{i}\pi\delta(x), \tag{75}$$

we get the following results.

For $|\lambda| = 1+\epsilon$, with $\epsilon\to 0^+$

$$\begin{aligned}\lim_{\epsilon\to 0^+}\widetilde{G}(\mathbf{r}\cdot\mathbf{n}, \mathbf{r}\cdot\mathbf{n}^\perp, \lambda) = G_+ &= \lim_{\epsilon\to 0^+}\frac{\mathrm{i}}{2\pi}\frac{1}{(\mathbf{r}\cdot\mathbf{n}^\perp)+\mathrm{i}\epsilon(\mathbf{r}\cdot\mathbf{n})}\\ &= \frac{\mathrm{i}}{2\pi}\mathrm{P.V.}\frac{1}{(\mathbf{r}\cdot\mathbf{n}^\perp)} + \frac{1}{2}\mathrm{sgn}(\mathbf{r}\cdot\mathbf{n})\delta(\mathbf{r}\cdot\mathbf{n}^\perp).\end{aligned} \tag{76}$$

For $|\lambda| = 1-\epsilon$, with $\epsilon\to 0^+$

$$\begin{aligned}\lim_{\epsilon\to 0^+}\widetilde{G}(\mathbf{r}\cdot\mathbf{n}, \mathbf{r}\cdot\mathbf{n}^\perp, \lambda) = G_- &= \lim_{\epsilon\to 0^+}\frac{\mathrm{i}}{2\pi}\frac{-1}{(\mathbf{r}\cdot\mathbf{n}^\perp)+\mathrm{i}\epsilon(\mathbf{r}\cdot\mathbf{n})}\\ &= -\frac{\mathrm{i}}{2\pi}\mathrm{P.V.}\frac{1}{(\mathbf{r}\cdot\mathbf{n}^\perp)} + \frac{1}{2}\mathrm{sgn}(\mathbf{r}\cdot\mathbf{n})\delta(\mathbf{r}\cdot\mathbf{n}^\perp).\end{aligned} \tag{77}$$

The sought Green's function being real, we can see that the real parts of the two limits, which are even functions of $(\mathbf{r}\cdot\mathbf{n}^\perp)$ (whereas the imaginary parts are odd functions of $(\mathbf{r}\cdot\mathbf{n}^\perp)$), are the same and give the expected answer.

3.2. Construction of solutions using the Green's functions

The Green's functions of equations (76) and (77), obtained in the last section for $\lambda\to 1$ with $|\lambda| > 1$ and with $|\lambda| < 1$, allow us to construct the limiting values $\Phi(\mathbf{r}, \lambda)$ of the solution of the equation

$$\nabla_\lambda\Phi(\mathbf{r}, \lambda) = \rho(\mathbf{r}). \tag{78}$$

Thus the two solutions $\Phi_\pm(\mathbf{r}, \mathbf{n})$ constructed respectively for $\lim_{\varepsilon\to 0^+}|\lambda| = 1+\varepsilon$ and for $\lim_{\varepsilon\to 0^+}|\lambda| = 1-\varepsilon$ are

$$\Phi_\pm(\mathbf{r}, \mathbf{n}) = \int_{\mathbb{R}^2}\mathrm{d}\mathbf{r}'G_\pm(\mathbf{r}-\mathbf{r}', \mathbf{n})\rho(\mathbf{r}') = \int_{\mathbb{R}^2}\mathrm{d}\mathbf{r}'G_\pm(\mathbf{r}', \mathbf{n})\rho(\mathbf{r}-\mathbf{r}'). \tag{79}$$

Now using the following parametrization:

$$\mathbf{r} = t\mathbf{n} + s\mathbf{n}^\perp \qquad \text{and} \qquad \mathbf{r}' = \tau\mathbf{n} + \sigma\mathbf{n}^\perp, \tag{80}$$

in equations (76) and (77) we get

$$\Phi_{\pm}(\mathbf{r},\mathbf{n}) = \frac{\pm 1}{2\pi\mathrm{i}}\left(\mathrm{P.V.}\int_{\mathbb{R}} \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}t \frac{\rho((\tau - t)\mathbf{n} + (\sigma - s)\mathbf{n}^{\perp})}{s} \right.$$
$$\left. \pm \mathrm{i}\pi \int_{\mathbb{R}} \mathrm{d}t\, \mathrm{sgn}(t)\rho((\tau - t)\mathbf{n} + \sigma\mathbf{n}^{\perp})\right). \tag{81}$$

The first integral on the right-hand side can be reformulated, using the definition of the Hilbert transform $\mathcal{H}f$ of a function f

$$(\mathcal{H}f)(\sigma) := \frac{1}{\pi}\mathrm{P.V.}\int_{\mathbb{R}} \mathrm{d}s \frac{f(s)}{\sigma - s}, \tag{82}$$

as

$$\Phi_{\pm}(\mathbf{r},\mathbf{n}) = \frac{\mp 1}{2\mathrm{i}}\mathcal{H}\int_{\mathbb{R}} \mathrm{d}t\, \rho(\mathbf{r} - t\mathbf{n}) + \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}t\, \mathrm{sgn}(t)\rho(\mathbf{r} - t\mathbf{n}). \tag{83}$$

The first term is the Hilbert transform of the Radon transform of ρ and the second term is just the Radon transform of ρ, the integration can be rewritten over all $\mathbb{R}$. Thus we have obtained the boundary values of two analytic functions in $\lambda \in \mathbb{C}$, at the boundary $|\lambda| = 1$. According to the general theory of representation of distributions of bounded support [15, 16], their jump across this boundary represents the true distribution $\varphi(\mathbf{r},\mathbf{n})$ on this boundary, which is

$$\varphi(\mathbf{r},\mathbf{n}) = \Phi_{+}(\mathbf{r},\mathbf{n}) - \Phi_{-}(\mathbf{r},\mathbf{n}) = -\mathrm{i}\mathcal{H}\int_{\mathbb{R}} \mathrm{d}t\, \rho(\mathbf{r} - t\mathbf{n}) = -\mathrm{i}(\mathcal{H}\mathcal{R}\rho)(\mathbf{r},\mathbf{n}), \tag{84}$$

where $\mathcal{R}\rho$ denotes the two-dimensional Radon transform of ρ. Since $\Phi_{\pm}(\mathbf{r},\mathbf{n})$ is a solution of equation (3), we also observe that

$$\mathbf{n}\cdot\nabla\varphi(\mathbf{r},\mathbf{n}) = 0. \tag{85}$$

Now an analytic function in the λ complex plane, which has exactly this jump on the unit circle T (or $|\lambda| = 1$), can be constructed by means of a Cauchy contour integral on T, as specified in [15, 16]

$$\Phi(\mathbf{r},\lambda) = \frac{1}{2\pi\mathrm{i}}\int_{T} \mathrm{d}\mu \frac{\varphi(\mathbf{r},\mu)}{\mu - \lambda}, \qquad \lambda \in \mathbb{C}\backslash T. \tag{86}$$

Equation (78) shows that, for $\lambda \to 0$

$$\rho(\mathbf{r}) = \lim_{\lambda\to 0} \lambda^{-1}\frac{\partial}{\partial \bar{z}}\Phi(\mathbf{r},\lambda), \tag{87}$$

also the Taylor expansion of equation (86), for $\lambda \to 0$, gives

$$\Phi(\mathbf{r},\lambda) = \frac{1}{2\pi\mathrm{i}}\int_{T} \mathrm{d}\mu \frac{\varphi(\mathbf{r},\mu)}{\mu} + \lambda\frac{1}{2\pi\mathrm{i}}\int_{T} \mathrm{d}\mu \frac{\varphi(\mathbf{r},\mu)}{\mu^2} + O(\lambda^2). \tag{88}$$

The first integral on the right-hand side of equation (88) is in fact equal to zero (see [17]) and equation (87) yields

$$\rho(\mathbf{r}) = \frac{1}{2\pi\mathrm{i}}\int_{T} \mathrm{d}\mu \frac{1}{\mu^2}\frac{\partial}{\partial\bar{z}}\varphi(\mathbf{r},\mu). \tag{89}$$

As $\mu \in T$, we set $\mu = \mathrm{e}^{\mathrm{i}\theta}$ and $\mathrm{d}\mu = \mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta$. This leads to

$$\rho(\mathbf{r}) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \mathrm{e}^{-\mathrm{i}\theta}\frac{\partial}{\partial\bar{z}}\varphi(\mathbf{r},\theta)$$
$$= \frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\theta(\cos\theta - \mathrm{i}\sin\theta)\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)\varphi(\mathbf{r},\theta). \tag{90}$$

So we have expressed the density $\rho(\mathbf{r})$ in terms of $\phi(\mathbf{r}, \theta)$, which is the value of the measured quantity on the boundary $\lambda = 1$. The question is to see whether this expression given by equation (90) is a Radon transform inversion formula [19]. Recalling that $\mathbf{n} \equiv (\cos\theta, \sin\theta)$ and $\mathbf{n}^{\perp} \equiv (-\sin\theta, \cos\theta)$, we can rewrite equation (90) as

$$\rho(\mathbf{r}) = \frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\theta[\mathbf{n}\cdot\nabla\varphi(\mathbf{r},\theta) + \mathbf{n}^{\perp}\cdot\nabla(\mathrm{i}\varphi)(\mathbf{r},\theta)], \tag{91}$$

because of equation (85), we end up with

$$\rho(\mathbf{r}) = \frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\theta\, \mathbf{n}^{\perp}\cdot\nabla(\mathrm{i}\varphi)(\mathbf{r},\theta). \tag{92}$$

Now, replacing φ by its expression from equation (84) and calling $g = \mathcal{R}\rho$, the Radon transform of $\rho(\mathbf{r})$, we arrive at the form

$$\rho(\mathbf{r}) = \rho(x, y) = -\frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\theta\, \mathbf{n}^{\perp}\cdot\nabla(\mathcal{H}g)(\mathbf{n}^{\perp}\cdot\mathbf{r},\theta). \tag{93}$$

Since $\mathbf{r} = (\mathbf{n}t + \mathbf{n}^{\perp}s)$, see equation (80), we have $\nabla = \left(\mathbf{n}\frac{\partial}{\partial t} + \mathbf{n}^{\perp}\frac{\partial}{\partial s}\right)$ and after insertion in equation (93) we obtain

$$\begin{aligned}\rho(\mathbf{r}) = \rho(x, y) &= -\frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\theta \frac{\partial}{\partial s}(\mathcal{H}g)(\mathbf{n}^{\perp}\cdot\mathbf{r},\theta)\\ &= -\frac{1}{4\pi^2}\int_0^{2\pi}\mathrm{d}\theta\,\mathrm{P.V.}\int_{-\infty}^{\infty}\mathrm{d}\sigma\frac{\frac{\partial g(\sigma,\theta)}{\partial\sigma}}{(\mathbf{n}^{\perp}\cdot\mathbf{r})-\sigma}.\end{aligned} \tag{94}$$

This gives precisely the standard inversion formula for the Radon transform, found in textbooks, e.g. [18, 19] and a remarkable connection of the directional derivative operator in two dimensions and the Radon transform has been established.

3.3. $\overline{\partial}$ method in complex analysis

In complex analysis, see e.g. [20], the non-homogeneous equation written with the operator of equation (63) has a standard solution. Here we show that $\rho(\mathbf{r})$ can be reconstructed using complex analysis. The $\mathbf{n}\cdot\nabla$ operator in the complex form yields the following non-holomorphic equation:

$$\frac{l}{\mathrm{i}}(|\lambda|^2 - |\lambda|^{-2})\frac{\partial}{\partial\overline{\zeta}}\Phi(\zeta) = \rho_*(X, Y), \tag{95}$$

where the real and imaginary parts of ζ are $X = \Re(\zeta)$ and $Y = \Im(\zeta)$, consequently, $\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta} = -2\mathrm{i}\,\mathrm{d}X\wedge\mathrm{d}Y := -2\mathrm{i}\,\mathrm{d}X\,\mathrm{d}Y$. If we use the definition of (61) for ζ then,

$$X = l(|\lambda| + |\lambda|^{-1})s \qquad \text{and} \qquad Y = l(|\lambda| - |\lambda|^{-1})t, \tag{96}$$

where $s = (-x\sin\theta + y\cos\theta)$ and $t = (x\cos\theta + y\sin\theta)$ are obtained from equation (80). Now, by definition

$$\widetilde{\rho}(X, Y) := \frac{\mathrm{i}}{l}\frac{\rho_*(X, Y)}{|\lambda|^2 - |\lambda|^{-2}}, \tag{97}$$

equation (95) is written as

$$\frac{\partial}{\partial\overline{\zeta}}\Phi(\zeta) = \widetilde{\rho}(X, Y). \tag{98}$$

Then, Φ is obtained by the following integral formula [21]:

$$\Phi(z_0) = \varphi(z_0) + \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{C}} \mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta}\frac{\widetilde{\rho}(X, Y)}{\zeta - z_0}, \tag{99}$$

where φ is the holomorphic solution of $\overline{\partial}$, i.e. $\frac{\partial}{\partial\overline{\zeta}}\varphi(\zeta) = 0$. Hence, $\varphi(z)$ is obtained by

$$\varphi(z_0) = \Phi(z_0) - \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{C}} \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}\frac{\widetilde{\rho}(X,Y)}{\zeta - z_0}. \tag{100}$$

The integral in equation (100) can be recast as

$$-\frac{1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}X\,\mathrm{d}Y\frac{\widetilde{\rho}(X,Y)}{X + \mathrm{i}Y - z_0}. \tag{101}$$

Using equations (97), (96) and $\mathrm{d}X\,\mathrm{d}Y = l^2(|\lambda|^2 - |\lambda|^{-2})\,\mathrm{d}s\,\mathrm{d}t$ the above integral becomes

$$\begin{aligned}&\frac{1}{\pi \mathrm{i}}\int_{\mathbb{R}}\int_{\mathbb{R}} l^2(|\lambda|^2 - |\lambda|^{-2})\,\mathrm{d}s\,\mathrm{d}t\frac{\frac{\mathrm{i}}{l}\frac{\rho_*(l(|\lambda|+|\lambda|^{-1})s,\,l(|\lambda|-|\lambda|^{-1})t)}{(|\lambda|^2-|\lambda|^{-2})}}{l[(|\lambda| + |\lambda|^{-1})s + \mathrm{i}(|\lambda| - |\lambda|^{-1})t - (\sigma + \mathrm{i}\tau)]}\\ &\quad = \frac{1}{\pi \mathrm{i}}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}s\,\mathrm{d}t\frac{\rho_*(l(|\lambda| + |\lambda|^{-1})s,\, l(|\lambda| - |\lambda|^{-1})t)}{(|\lambda| + |\lambda|^{-1})s + \mathrm{i}(|\lambda| - |\lambda|^{-1})t - (\sigma + \mathrm{i}\tau)},\end{aligned} \tag{102}$$

where $z_0 = l(\sigma + \mathrm{i}\tau)$. Now, by making the variable changes $(|\lambda|-|\lambda|^{-1})t' = ((|\lambda|-|\lambda|^{-1})t-\tau)$ and $(|\lambda| + |\lambda|^{-1})s' = ((|\lambda| + |\lambda|^{-1})s - \sigma)$ we obtain

$$\frac{1}{\pi \mathrm{i}}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}s'\,\mathrm{d}t'\frac{\rho(s' + \sigma, t' + \tau)}{\left(\frac{|\lambda|+|\lambda|^{-1}}{2}\right)s' + \mathrm{i}\left(\frac{|\lambda|-|\lambda|^{-1}}{2}\right)t'}, \tag{103}$$

where

$$\rho(s' + \sigma, t' + \tau) := 2\rho_*(l(|\lambda| + |\lambda|^{-1})s + l\sigma,\, l(|\lambda| - |\lambda|^{-1})t + l\tau). \tag{104}$$

We can rewrite $\rho(t'+\tau, s'+\sigma)$ with a vector argument in the form, i.e. $\rho((t'+\tau)\mathbf{n}+(s'+\sigma)\mathbf{n}^{\perp})$, then the integral in (103) becomes

$$\frac{1}{\pi \mathrm{i}}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}s'\,\mathrm{d}t'\frac{\rho\big((t' + \tau)\mathbf{n} + (s' + \sigma)\mathbf{n}^{\perp}\big)}{\left(\frac{|\lambda|+|\lambda|^{-1}}{2}\right)s' + \mathrm{i}\left(\frac{|\lambda|-|\lambda|^{-1}}{2}\right)t'}. \tag{105}$$

Finally, for $|\lambda| = 1$ then, we have

$$\begin{aligned}\frac{1}{\pi \mathrm{i}}\int_{\mathbb{R}} \mathrm{d}s'\frac{\int_{\mathbb{R}}\mathrm{d}t'\rho((t' + \tau)\mathbf{n} + (s' + \sigma)\mathbf{n}^{\perp})}{s'} &= \frac{\mathrm{i}}{\pi}\int_{\mathbb{R}} \mathrm{d}s'\frac{(\mathcal{R}\rho)(\tau\mathbf{n} + (s' + \sigma)\mathbf{n}^{\perp})}{-s'}\\ &= \mathrm{i}(\mathcal{H}\mathcal{R}\rho)(\tau\mathbf{n} + \sigma\mathbf{n}^{\perp}),\end{aligned} \tag{106}$$

where $(\mathcal{R}\rho) = \int_{\mathbb{R}}\mathrm{d}t'\rho((t'+\tau)\mathbf{n}+(s'+\sigma)\mathbf{n}^{\perp})$ is the two-dimensional Radon transform and the integral over s' is the Hilbert transform at point zero. Now, by substituting the above equation into equation (100), $\varphi(z_0)$ is obtained as

$$\varphi(z_0) = \Phi(z_0) - \mathrm{i}(\mathcal{H}\mathcal{R}\rho)(z_0). \tag{107}$$

Now the above equation and equation (86) yields a solution as a Cauchy integral (as a solution of a corresponding Riemann–Hilbert problem) on T, the unit circle $\{|\mu| = 1\}$

$$\begin{aligned}\Phi(\mathbf{r},\lambda) &= \frac{1}{2\pi \mathrm{i}}\int_T \mathrm{d}\mu\frac{\phi(\mathbf{r},\mu)}{\mu - \lambda}\\ &= \frac{1}{2\pi \mathrm{i}}\int_T \mathrm{d}\mu\frac{\Phi(\mathbf{r},\mu)}{\mu - \lambda} - \frac{1}{2\pi \mathrm{i}}\int_T \mathrm{d}\mu\frac{\mathrm{i}(\mathcal{H}\mathcal{R}\rho)(\mathbf{r},\mu)}{\mu - \lambda}.\end{aligned} \tag{108}$$

$\Phi(\mathbf{r},\mu)$ is identically zero on T [20]. Hence,

$$\Phi(\mathbf{r},\lambda) = -\frac{1}{2\pi \mathrm{i}}\int_T \mathrm{d}\mu\frac{\mathrm{i}(\mathcal{H}\mathcal{R}\rho)(\mathbf{r},\mu)}{\mu - \lambda}. \tag{109}$$

Consequently, by equation (87) we obtain

$$\rho(\mathbf{r}) = -\frac{\lambda^{-1}}{2\pi \mathrm{i}}\frac{\partial}{\partial\overline{z}}\int_T \mathrm{d}\mu\frac{\mathrm{i}(\mathcal{H}\mathcal{R}\rho)(\mathbf{r},\mu)}{\mu - \lambda}. \tag{110}$$

Finally, following the same procedure as described after equation (87), one can deduce equations (93) and (94).

4. Conclusion and perspectives

In this paper we have presented, in an elementary and tutorial way, the properties of a very simple inhomogeneous partial differential equation. The salient aspect is its connection to a class of integral transforms called divergent beam x-ray transform arising in reconstruction problems for x-ray computed tomography (CT). In fact, this equation is a special case of the stationary transport equation with attenuation and external source which has been investigated by many authors in two dimensions, in relation to the attenuated x-ray transform for ionizing radiation emission imaging [2]. This problem is several orders of magnitude harder than that treated here and has not been extended in higher dimensions. With our approach we hope to be able to deal directly with the three-dimensional case and not resort to a tomographic procedure for imaging object in real three dimensions. The two challenging problems are first the inclusion of two conditions to bring down the number of parameters to three and second the way to complexify the equation in $\mathbb{C}^2$.

Appendix A. On spherical bessel functions in $\mathbb{R}^N$

Spherical Bessel functions are described in many treatises, e.g. [22, 23]. This appendix is meant to explain rapidly their connection to ordinary Bessel functions. We start with the Helmholtz equation in $\mathbb{R}^N$

$$(\nabla^2 + k^2)\psi = 0. \tag{A.1}$$

The radial part of this equation is

$$\left(\frac{1}{r^{N-1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{N-1}\frac{\mathrm{d}}{\mathrm{d}r}\right) + k^2 - \frac{L^2}{r^2}\right)u_l(r) = 0. \tag{A.2}$$

By factorizing the solution into a radial part and an angular part, which is an N-dimensional harmonic function of level L such that $L^2 = l(l+N-2)$ and by making the following change of function

$$f_l(r) = r^{\frac{2-N}{2}}u_l(r), \tag{A.3}$$

we arrive at the following radial equation:

$$\frac{\mathrm{d}^2 f_l(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f_l(r)}{\mathrm{d}r} + \left(k^2 - \frac{\left(l+\frac{N-2}{2}\right)^2}{r^2}\right)f_l(r) = 0. \tag{A.4}$$

This is just the Bessel equation with index $\nu = l + \frac{N-2}{2}$. Thus, $u_l(r)$ has the following form:

$$u_l(r) = r^{\frac{2-N}{2}}J_{(l+\frac{N-2}{2})}(kr), \qquad u_l(r) = r^{\frac{2-N}{2}}N_{(l+\frac{N-2}{2})}(kr). \tag{A.5}$$

It is customary to denote the spherical Bessel functions by the symbol $j_l(x)$:

$$j_l(x) = \left(\frac{C_N}{x^{N-2}}\right)^{\frac{1}{2}}J_{(l+\frac{N-2}{2})}(x), \tag{A.6}$$

where C_N is a constant which is to be determined as follows.

Let $\mathbf{r} = (x_1, x_2, \ldots, x_N) = r\hat{\mathbf{x}}_N$, then

$$\delta(\mathbf{r}) = \frac{1}{\Omega_N r^{N-1}}\delta(r) = \int_0^\infty k^{N-1}\,\mathrm{d}k\int_0^\pi \Omega_{N-1}\sin\gamma\,\mathrm{d}\gamma\,\mathrm{e}^{2\mathrm{i}\pi kr\cos\gamma} \tag{A.7}$$

where $\Omega_N = 2\pi^{N/2}/\Gamma\left(\frac{N}{2}\right)$ is the 'solid' angle of N-dimensional hypersphere. Consequently, the spherically symmetric delta function in $\mathbb{R}^N$ has the form, generalizing the formulae found in for $N = 2, 3$

$$\frac{1}{\Omega_N r^{N-1}}\delta(r) = 2\Omega_{N-1}\int_0^\infty k^{N-1}\,\mathrm{d}k\, j_0(2\pi kr) = \frac{2\Omega_{N-1}}{(2\pi)^N}\int_0^\infty k'^{N-1}\,\mathrm{d}k'\, j_0(k'r), \tag{A.8}$$

thanks to the application of Hankel's identity for generalized spherical Bessel functions in $\mathbb{R}^N$

$$\frac{1}{\Omega_N r^{N-1}}\delta(r-r') = \frac{2\Omega_{N-1}}{(2\pi)^N}\int_0^\infty k'^{N-1}\,\mathrm{d}k'\, j_0(k'r)\, j_0(k'r'), \tag{A.9}$$

with $r'=0$. Now replacing the spherical Bessel function by their expression in terms of the Bessel function (equation (A.6)) we have

$$\begin{aligned}\frac{1}{\Omega_N r^{N-1}}\delta(r-r') &= \frac{2\Omega_{N-1}}{(2\pi)^N C_N r^{N-2}}\int_0^\infty k'\,\mathrm{d}k'\, J_{\frac{N-2}{2}}(k'r) J_{\frac{N-2}{2}}(k'r') \\ &= \frac{2\Omega_{N-1}}{(2\pi)^N C_N r^{N-1}}\delta(r-r'),\end{aligned} \tag{A.10}$$

hence,

$$C_N = \frac{2\Omega_{N-1}\Omega_N}{(2\pi)^N} = \frac{1}{2^{N-3}\sqrt{\pi}\,\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N-1}{2}\right)}. \tag{A.11}$$

Therefore, we obtain the following relation between $j_l(x)$ and $J_n(x)$:

$$j_l(x) = \left(\frac{2\Omega_{N-1}\Omega_N}{(2\pi)^N x^{N-2}}\right)^{\frac{1}{2}} J_{(l+\frac{N-2}{2})}(x) = \left(\frac{1}{2^{N-3}\sqrt{\pi}\,\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N-1}{2}\right)}\right)^{\frac{1}{2}} J_{(l+\frac{N-2}{2})}(x). \tag{A.12}$$

Having these expressions, a new form of the Hankel identity can be given in terms of these new Bessel functions.

Appendix B. The Green's function of the $\overline{\partial}$-operator in $\mathbb{R}^2$

This appendix is devoted to the derivation of the Green's function of the $\overline{\partial}$-operator in $\mathbb{C}$ from the Green's function of the Laplace operator in $\mathbb{R}^2$. The aim is to show that there is a problem to represent the delta function in $\mathbb{R}^2$, despite several investigations on this subject [8].

The Green's function of the Laplace operator is the solution of the equation

$$\triangle G(\mathbf{r}) = \delta(\mathbf{r}), \tag{B.1}$$

where $\delta(\mathbf{r})$ represents a unit point source at the coordinate origin. The problem has a manifest symmetry of revolution. Thus it is appropriate to use polar coordinates $\mathbf{r}=(r,\theta)$, so that

$$\triangle = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right] \tag{B.2}$$

$$G(\mathbf{r}) = G(r). \tag{B.3}$$

On the right-hand side, we have $\delta(\mathbf{r})$, but we shall not give yet its expression in terms of polar coordinates.

Inspection shows that $G(r)$ is of the form

$$G(r) = C\ln r + \text{const}. \tag{B.4}$$

To determine the coefficient C, we use the Stokes theorem. Both sides of equation (B.1) can be integrated over $\mathbb{R}^2$. Then the left-hand side may be treated as an integral on the divergence of ∇G, which is equal to the flux of ∇G through a closed curve around the origin. In Cartesian coordinates

$$(\nabla G)_x = G'(r)\cos\theta \tag{B.5}$$

$$(\nabla G)_y = G'(r)\sin\theta. \tag{B.6}$$

For a circle of radius r centered at the origin O, the area element d**a**, also in Cartesian coordinates, is

$$\mathrm{d}\mathbf{a} = (r\,\mathrm{d}\theta\cos\theta, r\,\mathrm{d}\theta\sin\theta), \tag{B.7}$$

so that the elementary flux is $G'(r)r\,\mathrm{d}\theta$, thus

$$\oint G'(r)r\,\mathrm{d}\theta = \oint \frac{C}{r}r\,\mathrm{d}\theta = 2\pi C. \tag{B.8}$$

On the right-hand side, the integration should yield

$$\int_{\mathbb{R}^2}\mathrm{d}\mathbf{r}\,\delta(\mathbf{r}) = 1. \tag{B.9}$$

We conclude that $C = 1/(2\pi)$ and

$$G(r) = \frac{1}{2\pi}\ln r + \text{const}. \tag{B.10}$$

We now shift to complex coordinates in $\mathbb{R}^2$ with $z = x + \mathrm{i}y$ and $\overline{z} = x - \mathrm{i}y$. We can see that the area element in $\mathbb{R}^2$ transforms as

$$\mathrm{d}x \wedge \mathrm{d}y = \frac{\mathrm{i}}{2}\mathrm{d}z \wedge \mathrm{d}\overline{z}, \tag{B.11}$$

one can check that this quantity is real. Now we also have

$$\triangle = 4\frac{\partial^2}{\partial z\partial\overline{z}}, \tag{B.12}$$

as well as

$$G(r) = \frac{1}{2\pi}\ln\sqrt{z\overline{z}} = \frac{1}{4\pi}\ln z\overline{z} = G(z,\overline{z}). \tag{B.13}$$

Thus equation (B.1) reads now

$$4\frac{\partial^2}{\partial z\partial\overline{z}}G(z,\overline{z}) = \delta(\mathbf{r}). \tag{B.14}$$

Hence one gets one of the two following possibilities:

$$\frac{\partial}{\partial\overline{z}}\frac{1}{\pi z} = \delta(\mathbf{r}) = \delta(x,y), \tag{B.15}$$

$$\frac{\partial}{\partial z}\frac{1}{\pi\overline{z}} = \delta(\mathbf{r}) = \delta(x,y). \tag{B.16}$$

We see that the explicit representation of the two-dimensional delta function in terms of z and $\overline{z}$ is irrelevant. It is sufficient to have for arguments of the two-dimensional delta function the real and imaginary parts of the complex variable utilized in the $\overline{\partial}$-operator.

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